International Journal of Inclusive Development
Citation IJID: **8**(01): 05-09, June 2022
DOI: 10.30954/2454-4132.1.2022.2
Peer-Reviewed Journal

An Analysis the Numerical and Differential Equation

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Received: 18 Mar., 2022 **Revised:** 27 May, 2022 **Accepted:** 04 June, 2022

Abstract

Remember that, in general, the word scalar is not restricted to real numbers. We are only using real numbers as scalars in this book, but eigenvalues are often complex numbers. Consider the square matrix A. We say that A is an eigen-value of A if there exists a non-zero vector x such that $Ax = \backslash x$. In this case, x is called an eigen-vector (corresponding to A), and the pair (A ,x) is called an eigen-pair for A. Therefore, A and x are an eigenvalue and an eigenvector, respectively, for A. Now that we have seen an eigen-value and an eigen-vector, let's talk a little more about them. Why did we require that an eigenvector not be zero? If the eigen-vector was zero, the equation $Ax = Xx$ would yield $0 = 0$. Since, this equation is always true, it is not an interesting case. Therefore, we define an eigen-vector to be a non-zero vector that satisfies $Ax = Xx$. However, as we showed in the previous example, an eigen-value can be zero without causing a problem. We usually say that x is an eigen-vector corresponding to the eigen-value A if they satisfy $Ax = Xx$. Since, each eigen-vector is associated with an eigen-value, we often refer to an x and A that correspond to one another as an eigen-pair. Did we notice that we called x "an" eigen-vector rather than "the" eigen-vector corresponding to A.

Keywords: Matrix, Complex, Square, Computation, Markov Chain

This is because any non-zero, scalar multiple of an eigenvector is also an eigenvector. If we let c represent a scalar, then we can prove this fact through the following steps. We have already computed eigenvectors in this course. When we studied Markov chains, we computed an eigenvector corresponding to A^r when we found the matrix to which the probabilities seemed to converge after many steps. Any row of that matrix is an eigenvector for A^T because all the rows of that matrix are the same. We write that row as a column vector when we use it as an eigenvector. The eigenvector that we found is called the dominant eigenvector.

The dominant eigenvector of a matrix is an eigenvector corresponding to the eigenvalue of largest magnitude (for real numbers, largest absolute value) of that matrix. Although we only found one eigenvector, we found a very important eigenvector. Many of the "real world" applications are primarily interested in the dominant eigenpair. The method that we used to find this eigenvector is called the power method. The power method will be explained later in this chapter. An eigenvector corresponding to the transpose of a transition matrix is the transpose of any row of the matrix that A^k converges to as k grows, these rows are all the same. The dominant eigenvalue is always 1 for

How to cite this article: Devi, C.N. (2022). An Analysis the Numerical and Differential Equation. *Int. J. of Inclusive Develop.*, **8**(01): 05-09.

Source of Support: None; **Conflict of Interest:** None

a transition matrix. The equation holds, so we have found an eigenpair corresponding to the transpose of the transition matrix.

For a transition matrix, the dominant eigenvalue is always 1. An eigenvector corresponding to $A - 1$ for A is the transpose of any row of A where A′ is the matrix to which A converges as k grows. The matrix for which we are finding an eigenpair must have been set up so that the columns (not the rows) add to 1 for the eigenvector to be read from A ; this is why we are dealing with A instead of A. These rules will require some modification if we are not dealing with a transition matrix.

If we do not have a transition matrix, can we still use the power method? Yes we can, but we need to modify the steps a bit because the dominant eigenvalue will not necessarily be the number one. Let us explain how to use the power method. An example follows the remarks to help clarify these steps.

- Let us choose a vector and call it x_0. Set $i = 0$.
- Multiply to get the next approximation for x using the formula x_M =AX,
- Divide every term in x_M by the last element of the vector and call the new vector
- Repeat steps 2 and 3 until and agree to the desired number of digits.

The vector obtained in step 4 is an approximate eigenvector corresponding to the dominant eigenvalue. We will call it x.

Because any constant multiple of an eigenvector is an eigenvector, we did not have to divide by the last element in the vector in step 3. We could have divided by any element or not divided at all. We divided so that our vector would not grow too large and we could tell when we had converged. We divided by the last element of the vector so that we would have a well-defined algorithm for using the power method. The choice of the last element over any of the others was arbitrary.

Therefore, if the last element is zero, divide by another element of the vector for that entire problem. When people program the power method on a computer, they usually divide by which is defined as the length of the vector, so that they don't have to worry about whether or not an element is zero.

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

Some calculators will not let we divide a vector by a constant. On those calculators, we can multiply by the multiplicative inverse (reciprocal) of the constant.

We will probably not be able to directly input the Rayleigh quotient into were calculator. It will consider the numerator and denominator as 1 by 1 matrices. We consider 1 by 1 matrices to be the same as real numbers, but were calculator may not consider them the same. Since we cannot divide matrices, were calculator will probably give we an error message.

Therefore, $x_2 = 1$ and $x_1 + 1\ (1) = 0$, so $x_x = -1$. This tells us that $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to = –4 when $A = \begin{bmatrix} 2 & 6 \\ 2 & -2 \end{bmatrix}$. Using the characteristic equation and Gaussian elimination, we are able to find both of the eigenvalues to the matrix and corresponding eigenvectors.

We can find eigenpairs for larger systems using this method, but the characteristic equation gets impossible to solve directly when the system gets too large. We could use approximations that get close to solving the characteristic equation, but there are better ways to find eigenpairs that we will study in the future. However, these two methods give we an idea of how to find eigenpairs.

Another matrix for which the power method will not work is the matrix $A = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$ because the eigenvalues are both the real number 5. The method that we showed we earlier will yield the eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to correspond to the eigenvalue A = 5. Other methods will reveal, and we can check, that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is also an eigenvector of A corresponding to A = 5. Notice that these two eigenvectors are not multiples of one another. If the same eigenvalue is repeated p times for a particular matrix, then there can be as many as p different eigenvectors that are

not multiples of each other that correspond to that eigenvalue.

We said that eigenvalues are often complex numbers. However, if the matrix *A* is symmetric, then the eigenvalues will always be real numbers. As we can see from the problems that we worked, eigenvalues can also be real when the matrix is not symmetric, but keep in mind that they are not guaranteed to be real.

Now that we know how to find eigenpairs, we might want to know what uses they have. The interesting uses come from larger systems, so we will just discuss them rather than solve them. Have we ever seen the video of the collapse of the Tacoma Narrows Bridge? The Tacoma Bridge was built in 1940. From the beginning, the bridge would form small waves like the surface of a body of water. This accidental behavior of the bridge brought many people who wanted to drive over this moving bridge. Most people thought that the bridge was safe despite the movement. However, about four months later, the oscillations (waves) became bigger. At one point, one edge of the road was 28 feet higher than the other edge. Finally, this bridge crashed into the water below. One explanation for the crash is that the oscillations of the bridge were caused by the frequency of the wind being too close to the natural frequency of the bridge. The natural frequency of the bridge is the eigenvalue of smallest magnitude of a system that models the bridge. This is why eigenvalues are very important to engineers when they analyze structures *(Differential Equations and Their Applications*, 1983, pp. 171-173).

DISCUSSION

The eigenvalue of smallest magnitude of a matrix is the same as the inverse (reciprocal) of the dominant eigenvalue of the inverse of the matrix. Since most applications of eigenvalues need the eigenvalue of smallest magnitude, the inverse matrix is often solved for its dominant eigenvalue. This is why the dominant eigenvalue is so important. Also, a bridge in Manchester, England collapsed in 1831 because of conflicts between frequencies. However, this time, the natural frequency of the bridge was matched by the frequency caused by soldiers marching in step. Large oscillations occurred and the bridge collapsed. This is why soldiers break cadence when crossing a bridge.

Frequencies are also used in electrical systems. When we tune the radio, we are changing the resonant frequency until it matches the frequency at which the station is broadcasting. Engineers used eigenvalues when they designed the radio. Frequencies are also vital in music performance. When instruments are tuned, their frequencies are matched. It is the frequency that determines what we hear as music. Although musicians do not study eigenvalues in order to play their instruments better, the study of eigenvalues can explain why certain sounds are pleasant to the ear while others sound "flat" or "sharp." When two people sing in harmony, the frequency of one voice is a constant multiple of the other. That is what makes the sounds pleasant. Eigen values can be used to explain many aspects of music from the initial design of the instrument to tuning and harmony during a performance.

Even the concert halls are analyzed so that every seat in the theater receives a high quality sound. Car designers analyze eigenvalues in order to damp out the noise so that the occupants have a quiet mode. Eigen value analysis is also used in the design of car stereo systems so that the sounds are directed correctly for the listening pleasure of the passengers and driver. When we see a car that vibrates because of the loud booming music, think of eigenvalues. Eigenvalue analysis can indicate what needs to be changed to reduce the vibration of the car due to the music.

Eigen values are not only used to explain natural occurrences, but also to discover new and better designs for the future. Some of the results are quite surprising. If we were asked to build the strongest column that we could to support the weight of a roof using only a specified amount of material, what shape would that column take? Most of us would build a cylinder like most other columns that we have seen. However, Steve Cox of Rice University and Michael Overton of New York University proved, based on the work of J. Keller and I. Tadjbakhsh, that the column would be stronger if it was largest at the top, middle, and bottom. At the points 1of the way from either end, the column could be smaller because the column would not naturally buckle there anyway.

This new design was discovered through the study of the eigenvalues of the system involving the column and the weight from above. Note that this

column would not be the strongest design if any significant pressure came from the side, but when a column supports a roof, the vast majority of the pressure comes directly from above.

Eigen values can also be used to test for cracks or deformities in a solid. Can we imagine if every inch of every beam used in construction had to be tested? The problem is not as time consuming when eigenvalues are used. When a beam is struck, its natural frequencies (eigenvalues) can be heard. If the beam "rings," then it is not flawed. A dull sound will result from a flawed beam because the flaw causes the eigenvalues to change. Sensitive machines can be used to "see" and "hear" eigenvalues more precisely.

Oil companies frequently use eigenvalue analysis to explore land for oil. Oil, dirt, and other substances all give rise to linear systems which have different eigenvalues, so eigenvalue analysis can give a good indication of where oil reserves are located. Oil companies place probes around a site to pick up the waves that result from a huge truck used to vibrate the ground. The waves are changed as they pass through the different substances in the ground. The analysis of these waves directs the oil companies to possible drilling sites.

There are many more uses for eigenvalues, but we only wanted to give we a sampling of their uses. When we study science or engineering in college, we will become quite familiar with eigenvalues and their uses. There are also numerical difficulties that can arise when data from real-world problems are used.

Chio Method for evaluating Determinants

Let us consider the determinant.

$$D = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix}$$

Let us suppose that some element, say c_4 is unity. If there is no element in the determinant, we can always divide a given row (or column) by one of its elements and take the element out in front of the determinant. We shall, however, for the moment assume that, as the determinant stands, $c_4 = 1$. Having chosen the pivotal element to be c_4, we make all the other elements in that row 0 by multiplying all members of the fourth column by *Cj* and subtracting the resulting members from those of the first column, by multiplying next by c_2 and subtracting the results from the second column, by multiplying this same fourth column of *D* by c_3 and subtracting from third column, and finally, by multiplying the fourth column by c_5 and subtracting from the fifth column. The result, since $c_4 = 1$, is

$$D = \begin{vmatrix} a_1 - c_1 a_4 & a_2 - c_2 a_4 & a_3 - c_3 a_4 & a_4 & a_5 - c_5 a_4 \\ b_1 - c_1 b_4 & b_2 - c_2 b_4 & b_3 - c_3 b_4 & b_4 & b_5 - c_5 b_4 \\ 0 & 0 & 0 & 1 & 0 \\ d_1 - c_1 d_4 & d_2 - c_2 d_4 & d_3 - c_3 d_4 & d_4 & d_5 - c_5 d_4 \end{vmatrix}$$

6j c, c_4 $6_2 c_2$ e_4 6_3 c_3 $e_{4c4} c_s c_{gc4}$ or by expanding by minors in terms of the elements of the third row,

$$D = \begin{vmatrix} a_1 - c_1 a_4 & a_2 - c_2 a_4 & a_3 - c_3 a_4 & a_5 - c_5 a_4 \\ b_1 - c_1 b_4 & b_2 - c_2 b_4 & b_3 - c_3 b_4 & b_5 - c_5 b_4 \\ d_1 - c_1 d_4 & d_2 - c_2 d_4 & d_3 - c_3 d_4 & d_5 - c_5 d_4 \\ e_1 - c_1 e_4 & e_2 - c_2 e_4 & e_3 - c_3 e_4 & e_5 - c_5 e_4 \end{vmatrix}$$

The fourth order terms are seen to be obtainable from the original determinant by following steps:

1. Choosing a pivotal element, such as c_4, which must either be equal to unity in advance or be made unity by division of the row (or column) by that element.
2. Crossing out the row and column belonging to this element.
3. Subtracting from each of the remaining elements the product of the elements found at the base of perpendiculars drawn from the element to the crossed out row and column.
4. Multiplying the determinant by $(-1)^{r+s}$, where *r* is the row and *s* is the column of the pivotal element.

This constitute Chio's rule, which can be applied repeatedly to reduce a given determinant eventually

to a first order determinant, i.e., to a number. Each application of the rule reduces the order by one.

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